Maxwell's Equations

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Unit Syllabus

Chapter-8: Electromagnetic Waves

Basic idea of displacement current, Electromagnetic waves, their characteristics, their transverse nature (qualitative ideas only).

Electromagnetic spectrum (radio waves, microwaves, infrared, visible, ultraviolet, X-rays, gamma rays) including elementary facts about their uses.

Module Wise Breakup of Unit Syllabus

Module 1	• Displacement current • Characteristics of em waves • Why are they called waves? • E and B vectors and relation between them • Transverse nature of em waves
Module2	• Explanation of em waves spectrum • Range of frequency , wavelength • Sources of em waves • Properties

	• Uses of em waves

Module 1

Introduction

Michael Faraday, from his experiments on electromagnetic Induction, found that a changing magnetic field can produce an electric field. Is the converse also true? That is can a changing electric field produce a magnetic field?

The answer was given by **James Clark Maxwell**, stating that the magnetic field is not only produced by electric current through a conducting wire, but also by a time varying electric field. Maxwell's statement was based on the concept of "displacement current", introduced by him, to rectify the logical inconsistency of Ampere's circuital law, applied to find the intensity of magnetic field (B) at a point, using two circular loops which lie just outside and just inside a capacitor, which is being charged by using a source of electricity.

Maxwell formulated a set of equations, which along with the Lorentz force formula explains, mathematically, all the basic laws in electromagnetic wave theory.

In this unit, we shall study displacement current, characteristics and properties of electromagnetic waves, including transverse nature.

Displacement Current

What is displacement current?Why do we need it?

In order to answer these and many other questions let us consider them in detail with a bit of historical background.

Maxwell highlighted a logical inconsistency of Amperes circuital law using the following illustration of charging of a capacitor.

Consider a parallel plate capacitor, with plates X and Y, connected to a battery B, as shown in the figure 1 (a). On pressing the key K, conduction current (i_c) starts flowing in the wires. The plates of the capacitor start storing charge. (The strength of i_c will exponentially decrease with continuous charging of the capacitor). Figure1 (a) shows a circular loop (l_1) of Radius R, centered on the wire carrying i_c, at any instant during the process of charging. P is a point on this circular loop (l_1).

To find the value of intensity of magnetic field (B) at P, Maxwell used Ampere's Law,

$$\oint_l \vec{B}.\vec{dl} = \mu_0 \times (current\ crossing\ the\ loop) \qquad (1)$$

$$L.H.S.\ of\ equation\ (1) = B \times 2\pi R \qquad (2)$$

$$R.H.S.\ of\ equation\ (1) = \mu_0 \times i_C \qquad (3)$$

Putting equations (2) and (3) in equation (1) we get the magnitude of $\vec{B}$ at P as:

$$B = \frac{\mu_0 i_C}{2\pi R} \qquad (4)$$

Now, consider a loop l_2, which coincides with the open mouth of a pot-shaped surface as shown in figure 1 (b) or a loop l_3, coinciding with the open mouth of a tiffin-box shaped surface as shown in figure 1 (c). You have to imagine that the plate X of the capacitor is enclosed in the surface considered.

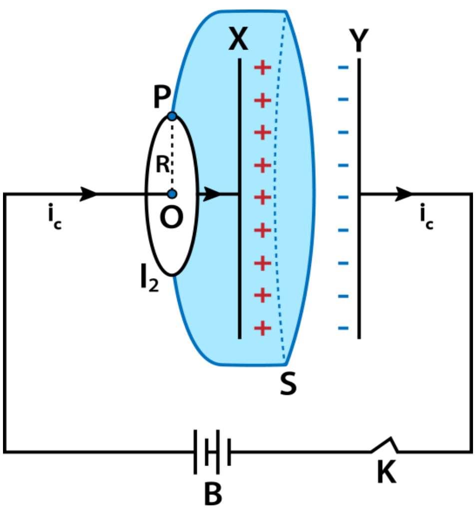


Figure 1 (b)

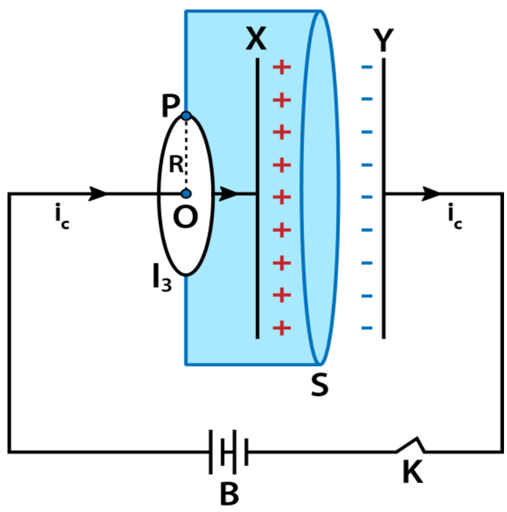


Figure 1(c)

For both these surfaces,

L.H.S of eq. (1), (that is, Ampere's circuital law), will still give the value B × 2πR.

But the R.H.S. will be $\mu_0 \times 0 = 0$, as there is no conduction current (i_c) passing through these surfaces.

B = 0, at the point P, if we find its value using the loops l_2 and l_3.

Therefore, it seems that, if we calculate the magnitude of B at P in one way, we get a finite value and in another method, we get zero. This clearly shows that Ampere's circuital law is logically inconsistent.

Clearly, something or other is missing from this law.

Maxwell thought that the missing part must have its origin in the electric field which is passing through the surface *S*, during charging. Obviously, this field is varying with time during the charging process.

Now, the magnitude of electric field, at any instant would be:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} \qquad (5)$$

Hence, the electric flux passing through the surface *S* is:

$$\phi_E = \frac{Q\,A}{A\,\varepsilon_0} = \frac{Q}{\varepsilon_0} \qquad (6)$$

$$\therefore \frac{d\,\phi_E}{dt} = \frac{d}{dt}\left(\frac{Q}{\varepsilon_0}\right) = \frac{1}{\varepsilon_0}\left(\frac{dQ}{dt}\right) = \frac{i}{\varepsilon_0} \qquad (7)$$

$$\text{or } i = \varepsilon_0 \frac{d\,\phi_E}{dt} \qquad (8)$$

But, as you know, there is no conduction current in between the plates of the capacitor. So, in equation ($i = \varepsilon_0 \frac{d\,\phi_E}{dt}$), 'i' is not conduction current and this current is produced by the time varying electric field in the space between the plates.

Maxwell called this current, formed by the varying electric field, displacement current (i_d):

$$\text{Hence, } i_d = \varepsilon_0 \frac{d\,\phi_E}{dt} \qquad (9)$$

It is clear that i_c and i_d are equal in magnitude, as

$$i_C = \frac{dQ}{dt} \text{ and } i_d = \varepsilon_0 \frac{d\,\phi_E}{dt} = \frac{dQ}{dt}$$

Also, $i_C = 0$ inside the capacitor and $i_d = 0$ in the connecting wires.

Hence, $i_c + i_d = i_c$ or i_d (depending on the position of the loops or location of point P)

Using the concept of displacement current (i_d), Maxwell established continuity of total current during charging, and modified Ampere's circuital law as following:

$$\oint_l \vec{B}.\vec{dl} = \mu_0 (i_C + i_d)$$

$$\text{or } \oint_l \vec{B}.\vec{dl} = \mu_0 \left(i_C + \varepsilon_0 \frac{d\,\phi_E}{dt}\right) \qquad (10)$$

Equation (10) is called Ampere - Maxwell law. This law predicts that magnetic fields can be produced by both the conduction current (i_c) and the displacement current i_d.

The Above Can be in Steps

i. Suppose the capacitor is an ideal capacitor, with a homogeneous electric field E between the plates and no electric field outside the plates.

ii. At a certain time t the charge on the capacitor plates is Q. If the plates have a surface area A then the electric field between the plates is equal to:

$$E = \frac{Q}{\varepsilon_0 A}$$

iii. The electric field outside the capacitor is equal to zero. The electric flux:

$$\phi_E = EA = \frac{Q}{\varepsilon_0}$$

iv. If a current I is flowing through the wire, then the charge on the capacitor plates will be time dependent. The electric flux will therefore also be time dependent, and the rate of change of electric flux is equal to:

$$\frac{d\phi_E}{dt} = \frac{1}{\varepsilon_0}\frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

v. The magnetic field around the wire can now be found by modifying Ampere's law:

$$\int \vec{B}.\vec{dl} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

vi. The effects of the varying electric flux and the electric current must be combined, and Ampere's law becomes:

$$\int \vec{B}.\vec{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

vii. Equation is frequently written as:

$$\int \vec{B}.\vec{dl} = \mu_0 (I + I_d)$$

where I_d is called **the displacement current** and is defined as:

$$I_d = \varepsilon_0 \frac{d\phi_E}{dt}$$

Think About These

- **Will there be displacement current, once the capacitor is charged after pressing key k?**
- **Will there be displacement current, once we remove the key k and allow the capacitor to discharge?**
- **What if we make and break the circuit using the key, will there be displacement current?**

- **What will be the duration and direction of displacement current if we connect an ac source instead of a dc battery?**
- **Would there be displacement current in an LC circuit at resonance?**

Magnetic Field Produced by Displacement Current

Since conduction current flows through a thin wire (mostly), the formula $B = \frac{\mu_0 i_C}{2\pi R}$ can be employed easily to find the value of B at any point near the wire.

Now, displacement current is produced by a varying electric field and the current flows through a large cross-section (A) in the field region, where A is taken perpendicular to the field direction. Hence, the value of B varies with distance r as illustrated in the figure 2 (a) and (b).

Let the plates X and Y of the parallel plate capacitor are circular in shape with radius R. Therefore, $A = \pi R^2$ is the area of the plates. Since the electric field $\vec{E}$ is crossing the entire area A, displacement current (i_d) produced due to variation in $\vec{E}$ will flow through an area A, with a displacement current density of

$$J_d = \frac{i_d}{\pi R^2} \qquad \ldots \qquad (11)$$

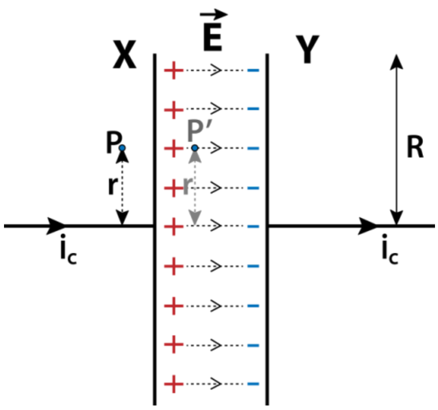


Figure 2(a)

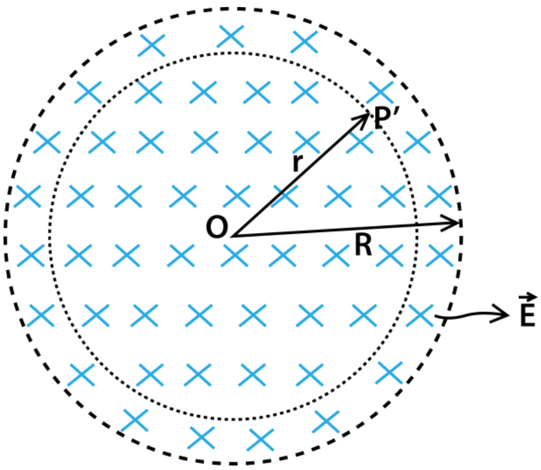


Figure 2(b)

Let P' be a point in this field $\vec{E}$, at a distance of r from the centre, as shown in the figure. The value of magnetic field intensity at P' will depend only on the displacement current flowing through an area $A' = \pi R^2$. This current i'_d is:

$$i'_d = J_d \times R^2$$

$= \frac{i_d}{\pi R^2} \times \pi r^2$

$= i_d \times \frac{r^2}{R^2}$ (12)

Therefore,

$B_{P'} = \frac{\mu_0 i'_d}{2\pi r}$

or $B_{P'} = \frac{\mu_0}{2\pi r} \times i_d \times \frac{r^2}{R^2}$

or $B_{P'} = \frac{\mu_0 i_d}{2\pi R} (\frac{r}{R})$ (13)

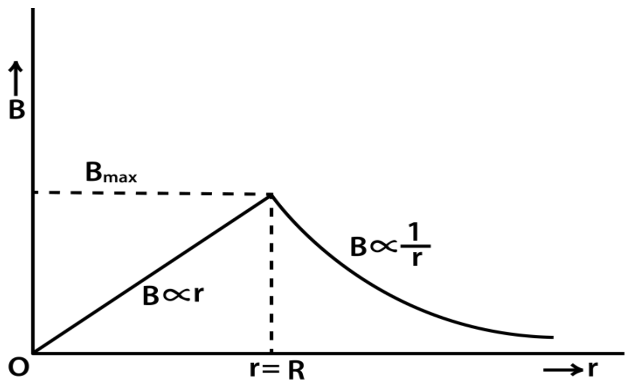
At $r = R$,

$B = B_{max} = \frac{\mu_0 i_d}{2\pi r}$ (14)

Therefore, eq. [$B_{P'} = \frac{\mu_0 i_d}{2\pi R} (\frac{r}{R})$] can be written as:

$B_{P'} = B_{max} (\frac{r}{R})$

This shows that $B_{P'} \propto r$ for $r \leq R$. Now, for any point for which $r > R$, the entire i_d will produce magnetic field which is given as



$B = \frac{\mu_0 i_d}{2\pi r}$ **, where** $r > R$**.** (15)

The variation of B with r in case of i_d can be represented as shown in the figure

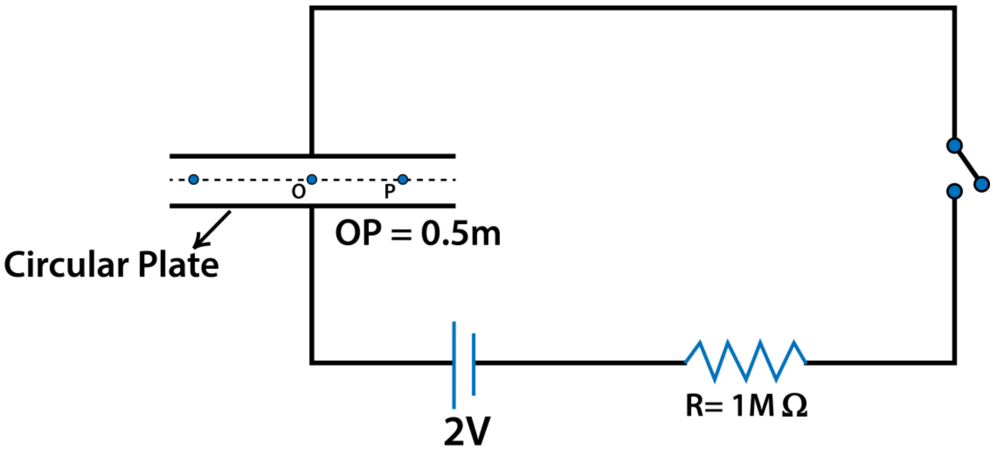
Solving - Problems - Using Displacement Current

Solved Example

A parallel plate capacitor with circular plates of radius 1 m has a capacitance of 1 nF.

At $t = 0$, it is connected for charging in series with a resistor R =1 MΩ, across a 2 volt battery.

Calculate the magnetic field at a point P, halfway between the centre and the periphery of the plates, after $t = 10^{-3}$ s. Use the expression for charge $q = CV(1 - e^{-t/\tau})$, where the time constant τ is equal to CR.



Solution

Given:

$R_a = 1\,m$

$OP = r = 0.5\,m$

$C = 1\,nF = 10^{-9}\,F$

$Resistance,\ R = 1\,M\Omega = 10^{6}\,\Omega$

$V = 10\,volts$

$\tau = CR = 10^{-9} \times 10^{6} = 10^{-3}\,s$

$CV = 10^{-9} \times 2 = 2 \times 10^{-9} C$

At time t, the charge (q) on the capacitor is:

$$q = CV\,(1 - e^{(\frac{t}{\tau})})$$

$$q = 2 \times 10^{-9}(1 - e^{(-\frac{t}{\tau})})\ C$$

The electric field (E) between the plates is:

$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{A\,\epsilon_0} = \frac{q}{\pi R_a^2 \epsilon_0} = \frac{q}{\pi\,\epsilon_0} \text{ as } A = 1$$

The electric flux through an area $A' = \pi r^2$ is:

$$\phi_E = E \times A' = \frac{q}{\pi\,\epsilon_0} \times \pi r^2 = \frac{q}{\epsilon_0} \times (\frac{1}{2})^2 = \frac{q}{4\,\epsilon_0}$$

Now, the displacement current through A' is:

$i'_d = \epsilon_0 \frac{d\phi_E}{dt} = \epsilon_0 \frac{d}{dt}\left(\frac{q}{4\epsilon_0}\right)$

$= \frac{1}{4} \frac{d}{dt}(q)$

$= \frac{1}{4} \frac{d}{dt}\left(2 \times 10^{-9}\left(1 - e^{(-\frac{t}{\tau})}\right)\right)$

$= \frac{1}{4} \times 2 \times 10^{-9} \times \frac{d}{dt}\left(- e^{(-\frac{t}{\tau})}\right)$

$= \frac{1}{2} \times 10^{-9} \times \frac{1}{\tau} \times \left(- e^{(-\frac{t}{\tau})}\right)$

$= \frac{1}{2} \times 10^{-9} \times \frac{1}{10^{-3}} \times e^{-1}$

$= \frac{1}{2} \times 10^{-6} \times \frac{1}{2.72}$

$= 0.5 \times 0.368 \times 10^{-6} A$

Therefore, the magnetic field (B) at P ($r = \frac{1}{2}$ m) is:

$B = \frac{\mu_0 i'_d}{2\pi r} = \frac{2 \times 10^{-7}}{\left(\frac{1}{2}\right)} \times 0.5 \times 0.368 \times 10^{-6} T$

$= 0.736 \times 10^{-13} T$

Solved Example

A capacitor is made of circular plates, each of radius 12 cm, separated by 5.0 mm. This capacitor is charged with a constant current of 0.15 A, using an external source.

(a) Calculate the rate of change of potential difference across the plates.

(b) Obtain the displacement current across the plates.

(c) Is Kirchhoff's rule applicable at each plate of the capacitor?

Solution

Given: $R = 12\, cm = 0.12\, m$

$d = 5.0\, mm = 5 \times 10^{-3}\, m$

$I = 0.15\, A$

$\epsilon_0 = 8.854 \times 10^{-12}\, C^2 N^{-1} m^{-2}$

Area of the plates, $A = \pi R^2 = 3.14 \times (0.12)^2\, m^2$

$C = \frac{A\epsilon_0}{d} = \frac{3.14 \times (0.12)^2 \times 8.854 \times 10^{-12}}{5 \times 10^{-3}}$

$= 80.1 \times 10^{-12} F$

a. $Q = CV$ and $I = \frac{dQ}{dt}$

$\frac{dQ}{dt} = C\frac{dV}{dt}$

$\therefore I = C\frac{dV}{dt}$

or $\frac{dV}{dt} = \frac{I}{C} = \frac{0.15}{80.1 \times 10^{-12}} = 1.87 \times 10^{9}\, Vs^{-1}$

b. The displacement current (i_d) is equal to the conduction current in the charging circuit (i_c)

$i_d = i_c$

$= 0.15\, A$

c. We know that the current in the circuit is:

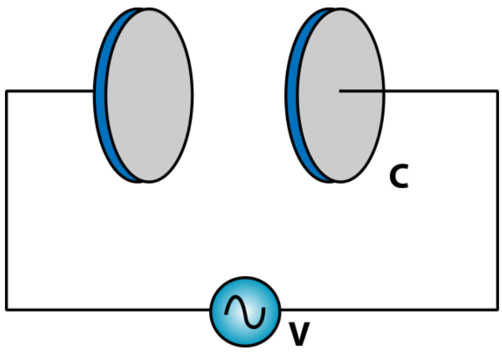
$i = i_c + i_d = i_c = i_d$

as where there is i_d, there is no i_c and vice versa. Hence the 'current' remains the same in the charging circuit, including the space between the plates.

Hence, total current flowing into the plate is equal to the total current flowing out of the plate, for both the plates. Therefore, Kirchhoff's law is true for the plates.

Solved Example

A parallel plate capacitor made of circular plates, each of radius 6.0 cm, has a capacitance of C = 100 pF. This capacitor is connected to a 230 volt a.c. supply with an angular frequency of =300 rad s^{-1}.



(a) What is the rms value of the conduction current?

(b) Is the conduction current equal to the displacement current?

(c) What is the amplitude of B at a point 3.0 cm from the axis between the plates?

Solution: Given:

$R = 6.0\, cm = 6 \times 10^{-2}\, m$

$$C = 100\,pF = 100 \times 10^{-12} F$$

$$\omega = 300\, rad\, s^{-1}$$

$$V = V_{rms} = 230\, volts$$

$$Capacitive\ reactance\ X_c = \frac{1}{\omega C}$$

a. rms value of conduction current is:

$$\left(i_C\right)_{rms} = I = \frac{V_{rms}}{X_c} = \frac{V}{\frac{1}{\omega C}} = \omega\, CV$$

$$= 300 \times 100 \times 10^{-12} \times 230$$

$$= 6.9 \times 10^{-6} A$$

$$= 6.9\ \mu A$$

b. Yes, $i_d = i_c$, even if the current is alternating or varying.

c. At $r = \frac{R}{2}$, the formula for magnetic field (B), $B = \frac{\mu_0 i_d}{2\pi R}\left(\frac{r}{R}\right)$ is valid even for alternating i_d.

When this i_d is maximum $\left(\sqrt{2}\left(i_d\right)_{rms}\right)$, the value of B also becomes maximum.

$$B_{max} = B_{peak} = \frac{\mu_0 \left(i_d\right)_{max}}{2\pi R}\left(\frac{r}{R}\right)$$

$$= \frac{2 \times 10^{-7} \times \sqrt{2}\left(i_d\right)_{rms}}{6 \times 10^{-2}}\left(\frac{3 \times 10^{-2}}{6 \times 10^{-2}}\right)$$

$$= \frac{\sqrt{2}}{6} \times 10^{-5} \times 6.9 \times 10^{-6}$$

$$= 1.63 \times 10^{-11}\ tesla$$

Consequences of Displacement Current

The concept of displacement current introduced by Maxwell to generalize Ampere's circuital law is of great significance, as it has remarkably established the symmetry between the laws of electricity and magnetism.

According to Faraday's Law of electromagnetic induction, the emf induced in a coil is equal to the rate of change of magnetic flux linked with the coil. But the existence of an emf clearly shows the existence of an electric field. The emf developed in a circuit or in a closed loop can be expressed in terms of work done per unit charge, mathematically as:

$\varepsilon = \oint \vec{E}.\vec{dl}$

Where ε = induced emf and $\vec{E}$ is the electric field.

According to Faraday's law of electromagnetic induction $= -\frac{d\phi_B}{dt}$.

Therefore, $\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$ (16)

This is one of the important equations used by Maxwell to explain his theory of electromagnetic waves. It may be noted here that: $\oint \vec{E}.\vec{dl}$ gives zero for static electric field, but it gives a finite non-zero value for time-varying electric field in space. The value of $\oint \vec{E}.\vec{dl}$ obtained for any closed loop (imaginary) in a time varying electric field is taken as the emf developed in that closed loop, and this emf is represented in the L.H.S. of eq. ($\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$). Now, if we can have a time varying magnetic field produced in space, somehow, then the R.H.S. of eq. ($\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$) will give a non-zero finite value and therefore the L.H.S. of this equation has to be finite and that means there will be a finite electric field. In short, eq. ($\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$) predicts, according to Maxwell, that a time-varying magnetic field in space can produce an electric field in that space.

Now, the modified Ampere's Law as given by eq. ($i_d = \varepsilon_0 \frac{d\,\phi_E}{dt}$) is:

$$\oint \vec{B}.\vec{dl} = \mu_0 \left(i_c + \epsilon_0 \frac{d\phi_E}{dt} \right)$$

Here, if we imagine a closed loop in space, there cannot be any i_c.

Therefore the above equation reduces to:

$$\oint \vec{B}.\vec{dl} = \mu_0 \left(\epsilon_0 \frac{d\phi_E}{dt} \right) \qquad (17)$$

This is another important equation used by Maxwell to explain his electromagnetic wave theory.

Eq. ($\oint \vec{B}.\vec{dl} = \mu_0 \left(\epsilon_0 \frac{d\phi_E}{dt} \right)$) predicts that if we produce a time varying electric field in space, $\frac{d\phi_E}{dt}$ will have a finite value (and so i_d will have a finite value and it will thus produce a magnetic field.

Now, the most fascinating aspect of Maxwell's explanations is that, if the magnetic field (B) produced by the time varying electric field (according to eq. ($\oint \vec{B}.\vec{dl} = \mu_0 \left(\epsilon_0 \frac{d\phi_E}{dt} \right)$) is also time varying, then $\frac{d\phi_B}{dt} \neq 0$ and thus we can get the electric field (E) produced back, according to equation ($\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$).

Maxwell predicted, further, that if this cycle happens, it will continue to happen, independently, by producing each other – that is the time varying electric field producing a time-varying magnetic field and this magnetic field producing back the electric.

These electric and magnetic fields, the variation in one producing the other, coupled together and travelling with the speed of light is called the electromagnetic wave.

Maxwell's Equations

To explain the production of electromagnetic wave,

Maxwell used eq.($\oint \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$) and eq. ($\oint \vec{B}.\vec{dl} = \mu_0 \left(\epsilon_0 \frac{d\phi_E}{dt} \right)$). But, to explain the various properties of these waves, Maxwell used two more fundamental equations from electricity and magnetism, which are Gauss's laws in these fields. So, there are, in all, four equations used to explain the electromagnetic wave theory. These are called Maxwell's equations. They are:

1. $\oint_S \vec{E}.\vec{dS} = \frac{charge\ enclosed\ by\ S}{\epsilon_0}$

2. $\oint_S \vec{B}.\vec{dS} = 0$ **(no monopole exists in magnetism)**

3. $\oint_l \vec{E}.\vec{dl} = -\frac{d\phi_B}{dt}$

4. $\oint_{l} \vec{B}.\ \vec{dl}\ =\ \mu_0\epsilon_0 \frac{d\phi_B}{dt}$ **(for space)**

Source of Electromagnetic Wave

Maxwell explained, theoretically, the production of EM waves with the help of eq. ($\oint \vec{E}.\vec{dl}\ =\ -\frac{d\phi_B}{dt}$) and ($\oint \vec{B}.\vec{dl}\ =\ \mu_0\left(\epsilon_0\frac{d\phi_E}{dt}\right)$)

The question arises that how do we produce the time varying electric field, in space, which can produce a time-varying magnetic field, through which the electric field is produced back? Stationary charges or charges moving uniformly cannot produce a time varying electric field. Thus, we require an accelerating electric charge to produce a time varying electric field, which can produce electromagnetic waves.

In practice, we basically use an oscillating electric charge (which is definitely accelerating) to produce a continuous train of electromagnetic waves, which was first demonstrated by Heinrich Rudolf Hertz in 1887. Thus, we understand that the source of EM waves is accelerating/oscillating charge and that as long as the charges continue to do this, electromagnetic waves are produced from those oscillations.

Summary

- We learnt that an electric current produces magnetic field and that two current-carrying wires exert a magnetic force on each other, we have seen that a magnetic field changing with time gives rise to an electric field
- James Clerk Maxwell (1831-1879), Electric field changing with time gives rise to a magnetic field
- Maxwell noticed an inconsistency in the Ampere's circuital law and suggested the existence of an additional current, called displacement current, to remove this inconsistency.
- Displacement current is due to time-varying electric field and is given by $i_C\ =\ \frac{dQ}{dt}$

  and $i_d\ =\ \varepsilon_0\frac{d\,\phi_E}{dt}\ =\ \frac{dQ}{dt}$
- Source of magnetic field in exactly the same way as conduction current.